B.Sc. Part III (Hons.) 5th Paper

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Metric space

Let X be a non-empty set.
A metric on X is a real function d of ordered pairs of elements of X which satisfies the following three conditions :

(a) $d(x,y) \geq 0$ and

$d(x,y) = 0 \Leftrightarrow x = y$

(b) $d(x,y) = d(y,x)$ [SYMMETRY]

(c) $d(x,y) \leq d(x,z) + d(z,y)$ [TRIANGLE INEQUALITY]

Here, the function d assigns to each pair (x,y) of elements of X a non-negative real number $d(x,y)$ which by symmetry does not depend on the order of the elements.

$d(x,y)$ is called the distance between x and y.

A metric space consists of two objects : a non-empty set X

and a metric d on X.

The elements of X are called the POINTS of the metric space (X, d).

In general, we denote the metric space $(X\ d)$ by the symbol X.

EXAMPLE 1

Let X be a non-empty set. We define d by

$$d(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$$

obviously d is a metric because it satisfies the conditions as shown below:

(a) $d(x,y) = 1$ when $x \neq y$ and $d(x,y)=0, x=y$

i.e. $d(x,y) \geq 0$ and

$d(x,y) = 0 \Leftrightarrow x = y$

(b) $d(x,y) = 1$

i.e distance between x and y = 1

i.e distance between y and x = 1

$\Rightarrow d(y,x) = 1$

$\Rightarrow d(x,y) = d(y,x)$

(c) we've to prove that

$$d(x,y) \leq d(x,z) + d(z,y)$$

If $x=y,\ y\neq z$ then $0 \leq 1+1$ which is true

If $x=y,\ y=z$ then $0 \leq 0+0$ " "

If $x\neq y,\ y=z$ then $1 \leq 1+0$ " "

If $x\neq y,\ y\neq z$ then $1 \leq 1+1$ " "

Thus, $d(x,y) \leq d(x,z)+d(z,y)$ for all possible choices x,y,z of X.

Here 'd' is called discrete/trivial metric.

2. USUAL METRIC ON R

Let R be the set of all real numbers then $d: R\times R \to R$ defined as

$$d(x,y) = |x-y| \ \forall\ x,y \in R$$

is a metric on R, known as usual metric on R.

Proof: (a) $d(x,y) = |x-y| \geq 0 \ \forall\ x,y \in R$

(b) $d(x,y)=0 \Leftrightarrow |x-y|=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$

(c) $d(x,y) = |x-y|$

Also, $d(y,x) = |y-x|$

$\because |x-y| = |y-x|$

$\Rightarrow d(x,y) = d(y,x)$

(d) $d(x,y) = |x-y| = |(x-z)+(z-y)|$

$\because |a+b| \leq |a|+|b|$

$\Rightarrow |(x-z)+(z-y)| \leq |x-z| + |z-y|$

$\Rightarrow d(x,y) \leq |x-z| + |z-y|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Hence, d satisfies all the conditions for a metric.